

Towards Resolving the Crab σ –Problem: A Linear Accelerator?

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ABSTRACT

Using the exact solution of the axisymmetric pulsar magnetosphere derived in a previous publication and the conservation laws of the associated MHD flow, we show that the Lorentz factor of the outflowing plasma increases linearly with distance from the light cylinder. Therefore, the ratio of the Poynting to particle energy flux, generically referred to as σ , decreases inversely proportional to distance, from a large value (typically $\gtrsim 10^4$) near the light cylinder to $\sigma \simeq 1$ at a transition distance R_{trans} . Beyond this distance the inertial effects of the outflowing plasma become important and the magnetic field geometry must deviate from the almost monopolar form it attains between R_{lc} and R_{trans} . We anticipate that this is achieved by collimation of the poloidal field lines toward the rotation axis, ensuring that the magnetic field pressure in the equatorial region will fall-off faster than $1/R^2$ (R being the cylindrical radius). This leads both to a value $\sigma = \sigma_s \ll 1$ at the nebular reverse shock at distance R_s ($R_s \gg R_{\text{trans}}$) and to a component of the flow perpendicular to the equatorial component, as required by observation. The presence of the strong shock at $R = R_s$ allows for the efficient conversion of kinetic energy into radiation. We speculate that the Crab pulsar is unique in requiring $\sigma_s \simeq 3 \times 10^{-3}$ because of its small translational velocity, which allowed for the shock distance R_s to grow to values $\gg R_{\text{trans}}$.

Subject headings: magnetic fields — MHD — pulsars: general

1. Introduction

The Crab nebula is certainly the best studied and possibly the most interesting of the supernova remnants. This is due to the fact that it has been detected at an extremely broad range of energies, from the radio to the TeV regime. What is of additional interest is that

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its expansion is powered by the energy input from the rapidly rotating pulsar PSR 0531+21 located at its center. As such it has been a laboratory for testing our models of pulsars, MHD winds, supernova remnants and radiation emission processes. The Crab is not unique in containing a pulsar at the center of a supernova remnant. However, it is unique in the efficiency of converting the power output associated with the pulsar spin-down to radiation, which reaches 20%, and to kinetic energy of the entire remnant which absorbs the remaining 80%.

Crucial in this efficient conversion of the pulsar power into radiation is thought to be the presence of a (reverse) strong shock at an angular distance of $10''$ (corresponding to a distance $\simeq 3 \times 10^{17}$ cm) from the location of the pulsar. This shock randomises the highly relativistic upstream MHD wind which is produced by the pulsar, thereby causing the wind to radiate away a major fraction of its available energy. The presence of this strong shock is predicated on the dominance of the relativistic MHD wind emanating from the pulsar by particles rather than magnetic field, i.e. that the magnetization parameter (defined below) at the shock distance has a value $\sigma_s \ll 1$.

The value of σ_s has been estimated in a variety of ways. Kennel & Coroniti (1984) (hereafter KC) have computed the detailed structure of the MHD flow downstream from the shock and concluded that matching the nebular expansion velocity at the nebular edge, using the low σ_s expansion of their solution, requires that $\sigma_s \simeq 3 \times 10^{-3}$. De Jager & Harding (1992) estimated the value of σ_s by fitting the spectrum and surface brightness of the nebula, under the assumption that the emission above 10 GeV is due to inverse Compton scattering of lower frequency photons, which presumably represent the synchrotron emission from the same electron distribution. Their estimate of σ_s and the radial distribution of the magnetic field are consistent with those proposed by KC.

However, as shown in Rees & Gunn (1974; eq. 1), matching the expansion velocity v_{ex} of the nebula at its edge at $R = R_N$ is just a statement of conservation of the momentum flux injected by the pulsar wind through an MHD shock at $R = R_s$, leading to $R_s/R_N \sim (v_{\text{ex}}/c)^{1/2}$, independent of the value of σ_s . KC showed that, if in addition $\sigma_s \ll 1$, one obtains $v_{\text{ex}}/c \sim \sigma_s$; however, the latter is not a condition necessary for matching the nebular expansion velocity to that of the MHD wind at R_s .

One is therefore led to the conclusion that the small value of σ_s associated with the Crab remnant is not a generic property of all remnants of similar morphology but specific to the Crab. Indeed, the Vela pulsar, located near the center of the Vela supernova remnant, has properties not too different from those of the Crab pulsar (other than its age), however, its non-thermal nebular emission is a much smaller fraction of the pulsar spin-down luminosity than it is in the Crab. The corresponding estimate for σ_s in the case

of Vela is $\sigma_s \simeq 1$, suggesting that prominent (as a fraction of the pulsar spin-down) nebular emission is generally associated with small values of σ_s which allow for the possibility of a strong MHD shock in the pulsar wind.

The values of σ_s inferred for the Vela and (even more for) the Crab pulsar MHD winds raise the following problem for these winds: the value of this parameter near the pulsar light cylinder is estimated to be quite high $\sigma \sim 10^{4-5}$ (Coroniti 1990 and references therein). Given that in a MHD wind $B_\phi \propto 1/R$ it is thought that both the magnetic and ram pressures should decrease like $1/R^2$, with their ratio thus remaining roughly constant at the value it attains near the light cylinder. Therefore, values $\sigma_s \sim 1$ (let alone $\sigma_s \sim 10^{-3}$) are hard to understand and yet overwhelmingly favored by observation.

A possible way out of this conundrum is to assume that the inertial component of the MHD wind is due to ions rather than leptons (electrons – positrons), leading to much smaller values of σ even near the light cylinder (Ruderman 1981; Arons 1983). However, one would then have to find a way of converting $\sim 20\%$ of the relativistic proton energy into relativistic electrons at the MHD shock.

This problem led to the suggestion that annihilation of magnetic field energy and conversion of the resulting energy into that of the outflowing particles could indeed provide for the required reduction in σ_s with distance (Coroniti 1990; Michel 1994). Such a solution is in principle possible (see though Lyubarskii & Kirk 2001), however, this process would work only on the magnetic dipole field component perpendicular to the direction of the pulsar angular velocity Ω . The component of the magnetic dipole field which is parallel to Ω is simply advected away with no possibility of such an annihilation. Since the observations (Aschenbach & Brinkmann 1975) seem to suggest that, at least for the Crab, the magnetic dipole is closely aligned with the pulsar rotation axis, it appears unlikely that a large fraction of the available magnetic energy could in fact annihilate.

However, in MHD flows, issues such as the asymptotic (or more generally the position dependent) value of σ are coupled to the global geometry of the flow. As shown by Heyvaerts & Norman (1989) for the non-relativistic case and by Chieueh, Li & Begelman (1991; hereafter CLB91) (see also Eichler 1993) for the relativistic one, these flows tend to asymptotically collimate; the associated divergence of lines from conical geometry could then also affect the corresponding value of σ through the “magnetic nozzling” which would convert magnetic energy to directed motion. It has been argued in the above references though, that because these flows collimate logarithmically in R such a “nozzling” is not observationally relevant for any plausible astrophysical situation. We also argue below, that the global geometry of the flow is indeed of consequence for the asymptotic value of σ , however our conclusions differ from those presently in the literature.

More recently, Chieueh, Li & Begelman (1998; hereafter CLB98) by an asymptotic analysis of the conservation and the perpendicular force balance (Grad-Safranov) equations, have argued for the implausibility of the transition of flows from high σ to low σ under axisymmetric, steady state conditions. To this end they considered a variety of plausible field geometries and argued for each of them that divergence of the magnetic field lines necessary to achieve a transition from $\sigma \gg 1$ to $\sigma \ll 1$ was incompatible with the balance of the corresponding pressures. While we believe their arguments to be sound we also think, as we argue later, that can also be circumvented.

Motivated by our recent exact solution of the axisymmetric pulsar magnetosphere (Contopoulos, Kazanas & Fendt 1999; hereafter CKF), we have decided to take a closer look at the problem of the entire MHD wind and its impact on the nebular morphology and dynamics. The solution of CKF provides the complete, global, magnetic field and associated electric current structure for an aligned rotator (Goldreich & Julian 1969) in the force free (i.e. with negligible inertia) MHD approximation, including their distribution across the crucial light cylinder surface. The main results of that paper are summarized below:

1. The magnetosphere consists of a region of closed field lines (dipole-like) extending up to the light cylinder, and a region of open field lines which cross the light cylinder and asymptote to a monopole-like geometry.
2. The magnetic field structure is continuous and smooth through the light cylinder, and thus, one cannot anymore invoke ‘dissipation zones’ at, or around, the light cylinder. In other words, one has to look elsewhere for the conversion of magnetic to particle energy and by consequence to the observed high energy radiation in pulsars.
3. A large scale electric current flows through the magnetosphere. The current distribution is uniquely determined by (a) the boundary condition at the origin (in our case a magnetic dipole), and (b) the requirement of no singularities at the light cylinder. The large scale electric circuit closes in an equatorial current sheet which connects to the edge of the polar cap³.
4. Outside the light cylinder, the solution with a dipole at the origin does not differ much from the well known monopole solution of Michel (1991).

³Note that this implies a discontinuity of the toroidal magnetic field component across the current sheet, which further leads to a discontinuity of the poloidal field component at the boundary of the closed field line region (‘dead zone’). This latter discontinuity might lead to magnetic field structure readjustments (as in the magnetar models of Duncan & Thompson 1992).

5. Finally, when we numerically checked whether the magnetic field structure obtained is capable to accelerate the flow of electrons and positrons from the polar cap, we obtained no significant acceleration. This last point merits special attention and will be revised, since, as we said, the observations suggest the presence of a hyper-relativistic wind of electrons and positrons at large distances.

In § 2 we outline in detail the so-called σ -problem using dimensional analysis of the corresponding MHD flow and indicate the arguments which could lead to its possible resolution. In § 3, using the MHD integrals of motion for the B-field geometry associated with the axisymmetric pulsar magnetosphere, we indicate the evolution of σ with radius and the eventual values it attains, providing an explicit resolution to the issue of its magnitude. Finally, in § 4 we consider other pulsar nebulae for which the value of σ_s has been estimated and our conclusions are drawn.

2. The σ problem

Since the Crab pulsar is believed to be an almost aligned rotator (Aschenbach & Brinkmann 1974), we will adopt the approximation of axisymmetry in our present discussion. Below, we provide a summary of our knowledge of axisymmetric pulsar magnetospheres based on CKF, using the Crab pulsar values as fiducial figures.

The field lines that cross the light cylinder emanate from a region near the pole, the polar cap, and are necessarily open. We calculate the polar cap radius⁴ to be equal to

$$R_{pc} = \sqrt{1.36} r_* \left(\frac{r_*}{R_{lc}} \right)^{1/2} = 0.9 \left(\frac{P}{33 \text{ ms}} \right)^{-1/2} \text{ km} \quad (1)$$

Here, $r_* = 10 \text{ km}$ is the canonical radius of a neutron star, and

$$R_{lc} = \frac{cP}{2\pi} = 1576 \left(\frac{P}{33 \text{ ms}} \right) \text{ km} \quad (2)$$

is the light cylinder radius (P the period of the neutron star rotation)⁵. Obviously, $R_{pc} \ll r_* \ll R_{lc}$. At the footpoints of the magnetic field lines on the polar cap, the

⁴As is obtained in CKF, $\Psi_{\text{open}} = 1.36\Psi_{\text{pc}}$, where Ψ_{open} and Ψ_{pc} are defined as the amount of magnetic flux crossing the distance to the light cylinder in the relativistic and nonrelativistic (i.e. undistorted) dipole solution respectively.

⁵Henceforth, we will denote cylindrical radii with capital R , and spherical radii with small r .

magnitude of the magnetic field B_* is of the order of 10^{12} G, the number density n_* of electrons/positrons in the outflowing wind is equal to

$$n_* = \kappa n_{GJ} \equiv \kappa \frac{B_*}{ePc} = 2 \times 10^{16} \left(\frac{\kappa}{10^4} \right) \left(\frac{B_*}{10^{12} \text{ G}} \right) \left(\frac{P}{33 \text{ ms}} \right)^{-1} \text{ cm}^{-3}, \quad (3)$$

and their Lorentz factor γ_* is of the order of 200 (see below). Here, e is the electron charge. The multiplicity coefficient κ expresses how many times the wind density surpasses the so called Goldreich-Julian density n_{GJ} at the base of the wind. The physics that determine κ and γ_* lie outside the context of ideal magnetohydrodynamics (CKF). In what follows, values of $\kappa \sim 10^{3-4}$ and $\gamma_* \sim 200$ are adopted from the cascade models of Daugherty & Harding (1982) who followed in detail cascades of high energy electrons in pulsar magnetospheres.

The open field lines contain an amount of magnetic flux

$$\Psi_{\text{open}} = \pi R_{pc}^2 B_* = 2.7 \times 10^{12} \left(\frac{P}{33 \text{ ms}} \right)^{-1} \left(\frac{B_*}{10^{12} \text{ G}} \right) \text{ G km}^2 \quad (4)$$

and an electron/positron wind with mass loss rate

$$\dot{M} = \pi R_{pc}^2 n_* m_e = 7.8 \times 10^{-31} \kappa \left(\frac{P}{33 \text{ ms}} \right)^{-2} \left(\frac{B_*}{10^{12} \text{ G}} \right) \text{ M}_{\odot} \text{ yr}^{-1} \quad (5)$$

from each polar cap (m_e is the electron rest mass). We have assumed here an almost uniform ‘loading’ of the polar cap field lines with matter. The wind carries a kinetic energy flux

$$W_{\text{Kinetic}} = \gamma \dot{M} c^3 = 7 \times 10^{-5} \kappa \left(\frac{\gamma}{200} \right) \left(\frac{P}{33 \text{ ms}} \right)^{-2} \left(\frac{B_*}{10^{12} \text{ G}} \right) L_{\odot}, \quad (6)$$

and the magnetic field carries a Poynting flux

$$\begin{aligned} W_{\text{Poynting}} &= \frac{\Omega}{2\pi c} \int_0^{\Psi_{\text{open}}} I(\Psi) d\Psi = \frac{\Omega}{2\pi c} f I \Psi_{\text{open}} \\ &= 10^4 f \left(\frac{I}{I_*} \right) \left(\frac{P}{33 \text{ ms}} \right)^{-4} \left(\frac{B_*}{10^{12} \text{ G}} \right)^2 L_{\odot} \end{aligned} \quad (7)$$

from each polar cap (Okamoto 1974). Here,

$$I_* \equiv \frac{\Omega \Psi_{\text{open}}}{2} \equiv \frac{1}{4} \cdot e n_{GJ} c \cdot \pi R_{pc}^2 \quad (8)$$

and I are the total amount of electric current flowing through the polar cap and the magnetosphere respectively. As we will see, contrary to \dot{M} and Ψ_{open} , I *cannot be a conserved quantity along the wind*. f is a factor of order unity which depends on the

distribution of the electric current $I(\Psi)$ across open field lines ($f = 0.67$ for the exact monopole solution of Michel (1991), and the numerical solution of CKF). The energy reservoir is obviously the neutron star spindown energy loss rate

$$W_{\text{Spindown}} = W_{\text{Kinetic}*} + W_{\text{Poynting}*} = W_{\text{Kinetic}} + W_{\text{Poynting}} \quad (9)$$

at all distances. The magnetization parameter σ is thus defined as

$$\sigma \equiv \frac{W_{\text{Poynting}}}{W_{\text{Kinetic}}} = \frac{\Psi_{\text{open}} I}{P c \gamma \dot{M} c^3} = \left(\frac{I}{I_*} \right) \left(\frac{\gamma_*}{\gamma} \right) \sigma_* . \quad (10)$$

Here,

$$\sigma_* \equiv \frac{W_{\text{Poynting}*}}{W_{\text{Kinetic}*}} = \frac{e f I_*}{\kappa \gamma_* m_e c^3} = \frac{1.6 \times 10^8}{\kappa} f \left(\frac{P}{33 \text{ms}} \right)^{-2} \quad (11)$$

is the value of the magnetization parameter near the surface of the neutron star.

As we discussed in the introduction, at a distance

$$r_s \sim 10^9 R_{lc} , \quad (12)$$

the pulsar wind slows down in a (reverse) shock, and approximately 20% of the neutron star spin-down luminosity is converted into optical to γ -ray radiation. The inferred value of the magnetization parameter is

$$\sigma_s \sim 3 \times 10^{-3} . \quad (13)$$

Using eq. (9) divided through with $W_{\text{Kinetic}*}$

$$\sigma_* + 1 = \frac{\gamma}{\gamma_*} (\sigma + 1) , \quad (14)$$

And making use of eq. (10), it is straightforward to see that the dramatic decrease in σ observed in the Crab could reasonably take place (based on whether the associated current is conserved or not) in two distinct regimes:

1. σ decreases from $\sigma_* \sim 10^4$ to $\sigma_{\text{trans}} = 1$. In that regime, the magnetospheric electric current is almost conserved, i.e.

$$\frac{I_{\text{trans}}}{I_*} = \frac{1}{2} , \quad (15)$$

whereas the wind Lorentz factor increases by a factor

$$\frac{\gamma_{\text{trans}}}{\gamma_*} = \frac{\sigma_*}{2} , \text{ i.e. } \gamma_{\text{trans}} = 3 \times 10^6 . \quad (16)$$

2. σ decreases from $\sigma_{\text{trans}} = 1$ to $\sigma_s = 3 \times 10^{-3}$. In that regime, the magnetospheric electric current decreases by a factor

$$\frac{I_s}{I_{\text{trans}}} = \frac{2}{\sigma_*}, \quad (17)$$

whereas the wind Lorentz factor remains almost constant, i.e.

$$\frac{\gamma_s}{\gamma_{\text{trans}}} = 2, \text{ i.e. } \gamma_s = 6 \times 10^6. \quad (18)$$

In fact, detailed modelling of the nebula's spectrum yields a shock upstream wind Lorentz factor $\gamma_s \sim 3 \times 10^6$, in agreement with the above (De Jager & Harding 1992).

Unfortunately, our theoretical understanding is not up to date with the above observational facts. The problem is that there is no indication for growth in the Lorentz factor in the inner magnetosphere, and the electric current is a conserved quantity in a force-free magnetosphere. Acceleration models invoquing dissipation zones near the light cylinder are not convincing anymore after CKF (e.g. Beskin, Gurevich & Istomin 1993), and models showing MHD acceleration at large scales define a-priori the field geometry (e.g. Takahashi & Shibata 1998). There exist also MHD models showing no acceleration at large scales, but this might have to do with their numerical extrapolation from small to large scales (e.g. Bogovalov & Tsinganos 1999, Bogovalov 2001). Our understanding is that current MHD acceleration models are incomplete.

Is it possible to account for the large scale acceleration of the pulsar wind in the context of ideal axisymmetric special-relativistic steady-state magnetohydrodynamics? We believe that the answer is yes, so let us now study the basic equations of the problem, focusing our analysis on the energy flux conservation equation along open field lines. As we will see, in order to reveal the flow acceleration, one has to be particularly careful when one takes limits of that equation at large distances.

3. The linear accelerator

Energy flux conservation implies that

$$\gamma \left(1 - \frac{R}{R_{lc}} \frac{v_\phi}{c} \right) = \gamma_* \quad (19)$$

along any open field line (e.g. Okamoto 1978, Mestel & Shibata 1994, Contopoulos 1995), with γ_* the initial value of the electron Lorentz factor ($\gamma_* \sim 200$ as discussed above). This

is just the differential form of the energy flux conservation equation (9). The induction equation further gives that

$$\frac{v_\phi}{c} = \frac{R}{R_{lc}} + \frac{v_p}{c} \frac{B_\phi}{B_p} . \quad (20)$$

In order to simplify the notation, we will concentrate our discussion on the last open field line along the equator.

As we argued above, in order to determine the evolution of the flow Lorentz factor γ with distance, it is reasonable to consider the two different distance regimes in the large scale pulsar magnetosphere we determined above (force-free, non force-free), and make different approximations in each of them. Let us first consider distances much larger than the light cylinder, where $\sigma \gtrsim 1$ (and most likely $\sigma \gg 1$). We show in the Appendix that, under force-free conditions (i.e. negligible inertia),

$$B_\phi = -\frac{R}{R_{lc}} B_p \quad (21)$$

when $R \gg R_{lc}$ (e.g. Okamoto 1997). This is identically valid in the analytical monopole solution (Michel 1991). It is also identically valid in the asymptotic monopole-like part of the more realistic solution with a dipole at the origin (CKF). Eqs. (19) and (20) then yield

$$\gamma \left[1 - \left(\frac{R}{R_{lc}} \right)^2 \left(1 - \frac{v_p}{c} \right) \right] = \gamma_* ,$$

which further yields

$$\gamma = \left[\gamma_*^2 + \left(\frac{R}{R_{lc}} \right)^2 \right]^{1/2} \rightarrow \frac{R}{R_{lc}} \quad (22)$$

for $R \gg R_{lc}$. This is a very important result. In addition to providing for the radial dependence of the wind's Lorentz factor, it also makes clear why CKF came (erroneously) to the conclusion that there is no acceleration across the light cylinder: as long as $R/R_{lc} < \gamma_* = 200$ (the value used by CKF based on the results of pair cascades in pulsars), the associated growth in γ is imperceptible (they would have found the increase had they chosen say $\gamma_* \sim 1$). The new result here is *the linear growth of the Lorentz factor γ with distance R* , for $R/R_{lc} \gg \gamma_*$. The reader can check that the pure monopole solution (Michel 1991) also shows this effect! This dependence indicates that the conversion of the flow energy from magnetic to kinetic is gradual. In fact, the conversion almost to equipartition takes place while the flow is essentially force-free and therefore our solution can be trusted. The main difference of our analysis with that of CLB98 is that upstream of their transition region the flow is essentially non-relativistic while ours has already a Lorentz factor $\gamma \gtrsim 10^5$. This constitutes the pivotal point in circumventing the analysis of

CLB98, who argued against a transition from magnetic to inertial dominance of the flow at large distances.

The linear growth, however, cannot continue beyond a distance

$$\gamma_* \sigma_* R_{lc} = 2R_{\text{trans}} = 3 \times 10^6 R_{lc} \ll r_s , \quad (23)$$

at which the Lorentz factor reaches the asymptotic value implied by mass conservation and the observed spin-down luminosity. The problem we are presented with has arisen from our neglect of matter in our assumption of negligible inertia (i.e. force-free) conditions. Note that eq. (21) is not valid at the distance where inertial and magnetic forces become comparable⁶. We must therefore proceed with caution through the energy conservation equation written as eq. (14). In the case of the Crab pulsar, we obtain

$$\frac{\sigma_s}{\sigma_{\text{trans}}} = \sigma_s = \frac{I_s}{I_{\text{trans}}} \frac{\gamma_{\text{trans}}}{\gamma_s} = \frac{1}{2} \frac{I_s}{I_{\text{trans}}} = \frac{1}{2} \frac{(RB_\phi)|_s}{(RB_\phi)|_{\text{trans}}} = 3 \times 10^{-3} \quad (24)$$

(see also Okamoto 1997 and references therein). The reader can check that the same result can also be obtained through the differential form of the Bernoulli equation [eq. 19]).

What does eq. (24) imply for the wind morphology at those distances? It is shown in the Appendix that, when $\sigma \ll 1$, $B_\phi \rightarrow -(R/R_{lc})B_p$, and thus

$$\frac{(R^2 B_p)|_s}{(R^2 B_p)|_{\text{trans}}} \sim 6 \times 10^{-3} . \quad (25)$$

In other words, beyond $R_{\text{trans}} \sim 5 \times 10^5 R_{lc}$, $B_p R^2$ does not remain constant but decreases with distance, and consequently, *field/flowlines should diverge away from monopolar geometry towards the axis of symmetry*. R in eq.(25) is the cylindrical radius, so assuming that B_p evolves roughly as $1/r^2$, this implies that, along a field/flowline,

$$(R/r)|_s \approx 10\% \cdot (R/r)|_{\text{trans}} , \quad (26)$$

or equivalently, a collimation by a factor of 10 in the cylindrical radius (the cylindrical radius at the shock spherical radius r_s will be about 10 times smaller than what it would be if the field/flow lines continued in the radial spherical direction from the transition distance).

⁶As we will see, eq. (21) becomes again asymptotically valid at the distances where inertial effects dominate. It is important to emphasize here that if we use eq. (21) in conjunction with eq. (19) in that latter regime, we will lose the effect of inertia, and will thus be led to the erroneous conclusion of unlimited linear growth in γ .

The precise field geometry can only be determined through a full solution of the Grad-Shafranov equation, whose solution we defer to a future publication. The degree of collimation implied by eq. (25) is not unreasonable, if one considers the Hubble Space Telescope (HST) observations of the Crab nebula in the optical, and the Chandra observations in X-rays. One can clearly see there the presence of a collimated polar flow, within a distance from the axis of symmetry of the order of ten times smaller than the overall size of the Crab nebula. Finally, for a much more detailed matching of the theoretical results to observation one should also consider the effects of the remnant into which the MHD wind is plowing. It is not obvious to the authors that every detail can be accounted for in terms of MHD winds and their self-collimation while ignoring the effects of the outlying medium.

Let us summarize our results for the Crab pulsar wind. The wind Lorentz factor grows linearly with distance up to a distance of the order of $10^6 \cdot R_{lc}$ where $\sigma \sim 1$. The wind/field geometry remains almost monopolar up to that distance. Beyond that, the wind collimates drastically towards the direction of the axis of symmetry. Its Lorentz factor remains close to its asymptotic value, and σ reaches its inferred value of 3×10^{-3} at the shock distance $10^9 R_{lc}$. Our present conclusion, namely the inevitable convergence of field/flowlines towards the axis of symmetry in order for the flow to accelerate to $\sigma \ll 1$, is not original (see Okamoto 1997 and references therein). Nevertheless, we are now in a position to make definite predictions about the degree of field/flow collimation, without solving the full non-force-free problem.

4. Conclusions, Discussion

We have presented above the spatial evolution of the kinetic energy associated with the wind from an axisymmetric pulsar magnetosphere. By solving the energy equation in the regions in which our exact solution of the MHD equations (CKF), based on the force-free assumption, is valid we indicated how the gradual acceleration of the expanding wind can lead to an equipartition between the magnetic and particle fluxes thus effecting the efficient conversion of magnetic to particle energy. This provides a first (to our knowledge) concrete example which exhibits such a conversion from Poynting to particle energy flux. We further indicated that at distances larger than that at which equipartition is established, the inertial effects should become important leading to a collimation of the wind and a further decrease in the ratio of magnetic to particle fluxes in agreement with HST/Chandra/CGRO observations of the Crab pulsar/nebula. Our work thus provides a straightforward resolution of the long standing σ -problem namely that of the particle over

the magnetic flux dominance of this object.

As discussed by Mestel (1999), the issue of the precise magnetospheric geometry and the associated evolution of the electron Lorentz factor is a coupled problem; he then raises the question of whether there may indeed be solutions in which the wind achieves its asymptotic value close to the light cylinder with the entire solution collimating at that distance but at the cost of requiring a domain in which dissipation enforces a local departure from the perfect conductivity condition. The work of CKF and our present argumentation indicates that this is not necessary and it is, in addition, compatible with the observations which require very little emission at radii smaller than r_s .

While the entire magnetospheric solution (including the inertial and external medium effects) is desirable in order to assess their mutual interactions and their effects on the precise geometry and flow dynamics, we believe that near (i.e. at $R \lesssim 10^4 R_{lc}$) the pulsar the flow geometry is so strongly dominated by its presence and the magnetic field, that the solution of CKF is essentially correct. As we argued earlier, we believe that important as the analysis of CLB98 is, its arguments could be circumvented simply because at the transition region the flow is almost in equipartition rather than magnetically dominated, as it is usually considered when analysing this situation (prior to our work the general assumption was that for a conical flow $\rho\gamma, B_\phi^2 \propto R^{-2}$, $B_\phi^2 \gg \rho\gamma$; efficient acceleration then required a very tightly wound-up B_ϕ^2 which would convert to kinetic density in a short distance; this is not any more necessary).

The results presented above are fairly general and could be applied to the case of other less well studied pulsar magnetospheres. Important parameters of our problem which are expected vary from pulsar to pulsar are the period P , the multiplicity coefficient κ , the initial flow Lorentz factor γ_* . Knowing P, κ, γ_* , one can estimate σ_* , and from it the characteristic distance

$$R_{\text{trans}} \sim \gamma_* \sigma_* R_{lc} \propto R_{lc} / \kappa \quad (27)$$

of the problem. Fortunately, not all these parameters are independent. The Lorentz factor of the pair resulting from the magnetospheric cascade γ_* as well as the multiplicity κ depend on the magnetic field B and the pulsar period P (Zhang & Harding 2000), with the multiplicity generally decreasing with decreasing B and increasing P , implying that there is in reality much less freedom in the problem. It would be of interest to compare the expected values of R_{trans} with the distance to the synchrotron nebula reverse shock R_s obtained from observations in other such nebulae. If R_{trans} is found to be $\ll R_s$, then our previous analysis applies, the flow should, like in the Crab, collimate towards the axis of symmetry, and at the same time σ should decrease to values $\ll 1$.

However, the Crab remnant seems to be a singular example, in that the neutron

star has remained very close to the center of the associated supernova remnant, and the synchrotron nebula has had enough time to grow to substantial distances. Furthermore, the entire nebula is powered exclusively by the pulsar, a fact not necessarily true with other remnants. The rather large peculiar velocities of these pulsars lead to geometries which are significantly affected by the pulsar motion and make similar comparisons difficult. The closest other remnant is that associated with the Vela pulsar for which the value of the magnetization parameter σ was estimated from detailed spectral fitting to be $\sigma \sim 1$ (de Jager et al. 1996). Similar conclusion was reached by Helfand et al. (2001) who analyzed the spatially resolved Chandra images of this source. It may therefore be that the low value of σ associated with the Crab nebula is specific to the conditions prevailing in this remnant.

The specific radial dependence of the pulsar wind's Lorentz factor is expected to have additional observational consequences concerning the emission of high energy radiation in systems containing pulsar winds. For example, Bogovalov & Aharonian (2000) computed the upComptonization of soft photons to TeV energies in the Crab through their interaction with the expanding MHD wind while Tavani & Arons (1997) and Ball & Kirk (2000) computed the corresponding radiation expected by the radio-pulsar Be star binary system PSR B1259-63 through the interaction of the relativistic wind with the photon field of the companion. We expect these predictions to be modified considerably in view of our present results. For example the above works assume that the wind achieves its asymptotic Lorentz factor shortly beyond the light cylinder, with the IC luminosity given by $L_{IC} \propto \tau(\gamma)L_s$ where L_s is the soft photon luminosity and $\tau(\gamma)$ is the optical depth for scattering by electrons of Lorentz factor γ . Clearly, our proposed linear evolution of the wind Lorentz factor γ would lead to a very different dependence for the optical depth of electrons with a given Lorentz factor in the expanding wind. This in turn should lead to a high energy gamma ray spectrum of very different form than that obtained under the assumption of constant electron Lorentz factor used in the aforementioned works. We expect that careful modeling of the resulting spectra and comparison to (future) high energy γ -ray observations will allow to confirm or disprove the proposed linear with r evolution of the wind Lorentz factor.

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Appendix

We derive here simple relations between the toroidal and poloidal magnetic field components at large distances from the light cylinder.

We will first consider the regime where force-free conditions are valid ($\sigma \geq 1$), and according to CKF, the field has attained a monopolar distribution, where all physical quantities become functions of the spherical angle θ . In that regime, the pulsar equation takes the simple form (eq. 15 in CKF)

$$\frac{d^2\Psi}{dt^2} = -\frac{d\Psi}{dt} \frac{1+2t^2}{t(1+t^2)} + \frac{4R_{lc}^2 I \frac{dI}{d\Psi}}{c^2 t^2 (1+t^2)} , \quad (28)$$

where, $t \equiv \tan \theta$, with boundary conditions $\Psi(t=0) = 0$, and $\Psi(t=\infty) = \Psi_{\text{open}}$. Bearing in mind that $I(\Psi=0) = 0$ (i.e. no singular current along the axis of symmetry), eq. (28) can be integrated to yield

$$I = -\frac{c}{2R_{lc}} \frac{d\Psi}{d\theta} \sin \theta \quad (29)$$

(the reader can check that I and B_p point in opposite directions, thus the minus sign). It is now straightforward to see that

$$\frac{B_\phi}{B_p} \equiv \frac{\frac{cI}{2R}}{-\frac{1}{r^2 \sin \theta} \frac{d\Psi}{d\theta}} = -\frac{R}{R_{lc}} . \quad (30)$$

This is a very general result, which does not depend on whether we have a dipole or a (split) monopole at the center (the reader can check that eq. (30) is directly satisfied for the Michel 1991 split monopole solution).

We will next consider the asymptotic regime where force-free conditions are not valid anymore ($\sigma \ll 1$, or $4\pi\rho\gamma v_p^2/B_p^2 \gg 1$, where ρ is the matter density in the observer's fixed frame). In that regime, we need to keep all the terms in the expression for B_ϕ (e.g. Contopoulos 1994)

$$B_\phi = \frac{\frac{cI}{2R} - \frac{4\pi\rho\gamma v_p}{B_p} R\Omega}{1 - \frac{4\pi\rho\gamma v_p^2}{B_p^2}} \rightarrow -\frac{\frac{cI}{2R} - \frac{4\pi\rho\gamma c}{B_p} R\Omega}{\frac{4\pi\rho\gamma c^2}{B_p^2}} \rightarrow \frac{R}{R_{lc}} B_p . \quad (31)$$

The reader should keep in mind that, in order to obtain eq. (31), we have taken a limit, which is not the case for eq. (30).